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Exponentially fitted non-polynomial cubic spline method for time-fractional singularly perturbed convection-diffusion problems involving large temporal lag

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Abstract

Objective The main purpose of this work is to present an exponentially fitted non-polynomial cubic spline method for solving time-fractional singularly perturbed convection-diffusion problem involving large temporal lag.

Result The time-fractional derivative is considered in the Caputo sense and discretized using backward Euler technique. Then, on uniform mesh discretization, a non-polynomial cubic spline scheme is constructed along the spatial direction. To alleviate the effect of the perturbation parameter, an exponential fitting factor is introduced to the scheme. The parameter-uniform convergence of the proposed method is proved rigorously and shown to be ε-uniform convergent with order of convergence $O((\Delta t)^{2-\alpha} + M^{-1})$. The validity of the proposed method is tested using model examples and the experimental results are in agreement with the theoretical expectation and produces more accurate solution than some existing methods in the literature.

Keywords Time-fractional, Exponentially fitting factor, Non-polynomial, Cubic spline, Caputo derivative

Introduction

Even though the history of fractional calculus trace back to 1695, when Leibnitz introduced for the first time, it doesn't applied in the modeling of problems arising in science and engineering for a long time. However, for the last few decades, fractional calculus began to attract the increasing attention of many scientist and researchers due to its wide application in the modeling of real life problem. This is due to the fact that fractional differential equations rather than integer order differential equations can better model natural physics process and dynamic system processes [1–3]. Fractional differential equation is the generalization of classical order differential equation by replacing the integer order derivative with arbitrary fractional order [3]. Finance, hydrology, control system, viscoelasticity, damping laws, fluid mechanics, biology, physics, engineering, modeling of earth quakes and etc are some of its application areas, [1, 2] and the references therein.

In general, fractional partial differential equations(FPDEs) can be divided as time-fractional partial differential equations, space-fractional partial differential equations or space-time fractional partial differential equations [4]. The analytical solution of many time-fractional partial differential equations are not available due to their difficulties in solving such differential equations exactly and even if the analytical solution is available, their construction with special functions make their computations very difficult. Thus, the numerical techniques has gained a great keenness in solving such equations numerically [5, 6]. Crank-Nicholson method

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based on spline functions with a tension factor [1], compact finite difference method [5], finite difference method [6, 7], a collocation method based on cubic-trigonometric B-splines approach [8], a fully implicit finite difference scheme based on extended cubic B-splines [9] are few of the recently developed numerical methods to solve time-fractional partial differential equations.

Sometimes, the future state of a certain physical problems may not only determined by their current state, but also by their past history and such physical problems are modeled by a delayed partial differential equations. For example, the time to maturity and incubation time, delayed feedback, time to transport, population dynamics, the time lag for getting information and HIV infection of $CD4+T$-cells to describe the time between infection of $CD4+T$-cells and the emission of viral particles on a cellular level [1, 10] are few of the application areas of delayed partial differential equations. Many considerable works have been carried out on the numerical methods for solving time-fractional delay partial differential equations [5, 10, 11].

In this paper, on the rectangular domain $D=\Omega_x\times\Omega_t$, we have considered the time-fractional singularly perturbed convection-diffusion problem involving large temporal lag of the form:

$$\begin{cases} \mathcal{L}_\varepsilon u(x,t)\equiv D_t^\alpha u(x,t)-\varepsilon u_{xx}(x,t)+p(x)u_x(x,t)+q(x,t)u(x,t)=-r(x,t)u(x,t-\wp)+\\ \qquad\qquad g(x,t),\quad (x,t)\in D,\\ u(x,t)=\psi(x,t),\ \text{for}\ (x,t)\in\Gamma_b=[0,1]\times[-\wp,0],\\ u(0,t)=\phi_l(t),\ \text{for}\ (x,t)\in\Gamma_l=\{0\}\times(0,T]=\{(0,t):0<t\le T\},\\ u(1,t)=\phi_r(t),\ \text{for}\ (x,t)\in\Gamma_r=\{1\}\times(0,T]=\{(1,t):0<t\le T\}, \end{cases} \tag{1}$$

where $0<\alpha<1$, $\Gamma=\Gamma_l\cup\Gamma_r\cup\Gamma_b$ is the boundary, $D=\Omega_x\times\Omega_t=(0,1)\times(0,T]$, D_t^α is the Caputo fractional derivative, $\wp$ is a delay parameter and ε is a positive constant satisfying $0<\varepsilon\ll 1$ called singular perturbation parameter. If $p(x)\ge p>0$, $q(x,t)\ge q>0$, $r(x,t)\ne 0$ and $g(x,t)$ are smooth and bounded functions on the domain $\overline{D}$ and the given initial data and boundary conditions are also smooth and bounded in their domain, then the solution of the model problem (1) exhibit a right boundary layer of width $O(\varepsilon)$. When $\alpha=1$, the problem in (1) gives the usual integer order singularly perturbed convection-diffusion problem. The numerical treatment of such classical or integer order problem with a delay and without a delay have been studied extensively by many researchers (see [12–19] and the references therein).

Unlike the classical order or integer order singularly perturbed partial differential equations, the fractional order in particular the time-fractional singularly perturbed partial differential equations are not studied well and needs attention. In such problems, due to the presence of the singular perturbation parameter ε, all classical numerical methods that are used to solve time-fractional PDEs fails or deteriorate to solve such problems. Moreover, the presence of the fractional order in the given differential equation is also another challenge in solving such problems. To the best of the author's knowledge, a stable finite difference method [20], cubic B-spline collocation method [21] and nonstandard finite difference method [22] are the only recently proposed numerical methods for solving the problem under consideration. As a result, it is possible to say that the numerical treatment of the considered problem is at infant stage. Motivated by the aforementioned gap, we have proposed a non polynomial cubic spline numerical scheme for the problem under consideration. To develop the scheme, we have considered the time-fractional derivative in the Caputo sense and discretized using implicit Euler method. The Caputo fractional derivative allows us to use the classical initial and boundary conditions. Moreover, it takes account of the interaction within the past. Then, a non polynomial cubic spline scheme is constructed along a uniform spatial discretization.

Some preliminaries and properties of continuous solution

Firstly, we present some basic definitions for fractional derivatives which are used here with this paper.

Definition 1 Let z be a complex number with $(\mathfrak{Re}(z))>0$. Then, the function defined by:

$$\Gamma(z)=\int_0^\infty e^{-\xi}\xi^{z-1}d\xi,$$

is called gamma function.

Definition 2 (Ref. [20]) The α-order Caputo fractional differentiation of a function $u(x,t)$ with respect to t is defined by:

$$\frac{\partial^\alpha u(x,t)}{\partial t^\alpha}=\begin{cases}\frac{1}{\Gamma(n-\alpha)}\int_0^t\frac{\partial^n u(x,s)}{\partial s^n}(t-s)^{n-\alpha-1}ds, & \text{if}\quad \alpha\in(n-1,n),\\ \frac{\partial^n u(x,t)}{\partial t^n} & \text{if}\quad \alpha=n.\end{cases}$$

The differential operator, $\mathcal{L}_\varepsilon$ in (1) satisfies the following continuous maximum principle.

Lemma 1 [22] *Let the function $\vartheta(x,t) \in C^2(D) \cap C^0(\overline{D})$ satisfies $\vartheta(x,t) \geq 0$, for $(x,t) \in \Gamma$ and $\mathcal{L}_\varepsilon \vartheta(x,t) \geq 0$, $\forall (x,t) \in D$. Then, $\vartheta(x,t) \geq 0, \forall (x,t) \in \overline{D}$.*

The stability of the operator $\mathcal{L}_\varepsilon$ and the ε-uniform boundedness for the solution of (1) is given by the following lemma.

Lemma 2 *The ε-uniform bound on the solution $u(x, t)$ of the continuous problem (1) satisfy*:

$$\|u\| \leq \|u\|_\Gamma + \frac{||\mathcal{L}_\varepsilon u||}{q},$$

where $\|u\|_\Gamma$ is the boundedness of the solution on $\Gamma = \Gamma_l \cup \Gamma_r \cup \Gamma_b$.

Proof By defining the barrier functions:

$$\vartheta(x,t) = \|u\|_\Gamma + \frac{||\mathcal{L}_\varepsilon u||}{q} \pm u(x,t), \quad (x,t) \in \overline{D}.$$

and using lemma 1 ends the proof. □

The numerical method

Temporal discretization

Firstly, the temporal domain $[0, T]$ is discretized uniformly with step size Δt as $\Omega_t = \{t_j = j\Delta t, \quad j = 0,1,2,3,...,N, \quad \Delta t = \frac{T}{N}\}$ and $\Omega_\wp^n = \{j\Delta t, \quad j = 0,1,2,3,...n, t_n = \wp, \Delta t = \frac{\wp}{n}\}$, where N is the number of mesh points in the time interval $[0, T]$ which is chosen in such a way that $\wp = n\Delta t$ for some positive integer $n \in (0, N)$. Then, the time- fractional derivative term of (1) is considered in the Caputo sense and at $t = t_{j+1}$, it is approximated by the following quadrature formula:

$$D_t^\alpha u(x,t_{j+1}) = \frac{1}{\Gamma(1-\alpha)} \int_0^{t_{j+1}} \frac{\partial u(x,\tau)}{\partial \tau} (t_{j+1} - \tau)^{-\tau} d\tau.$$

Then, following the approach in [21], the Caputo fractional derivative $D_t^\alpha u(x,t)$ at (x, t_{j+1}) is approximated by:

$$D_t^\alpha u(x,t_{j+1}) = \beta \sum_{k=0}^{j} b_k \Big(u(x,t_{j-k+1}) - u(x,t_{j-k}) \Big) + e_{\Delta t}^{j+1}. \tag{2}$$

where $\beta = \frac{1}{(\Delta t)^\alpha \Gamma(2-\alpha)}$, $b_k = (k+1)^{1-\alpha} - (k)^{1-\alpha}$ and $e_{\Delta t}^{j+1} = \frac{O(\Delta t)}{\Gamma(2-\alpha)} \sum_{k=0}^{j} \int_{t_k}^{t_{k+1}} (t_{j+1} - \tau) d\tau$.

Now, the application of (2) into (1) gives the semi-discrete problem:

$$\begin{cases} \left(\beta + \mathcal{L}_\varepsilon^{\Delta t}\right) U^{j+1}(x) = R^{j+1}(x), \\ U^{j+1}(x) = \psi^{j+1}(x), \text{for} \quad (x,t_{j+1}) \in [0,1] \times [-\wp, 0], \\ U^{j+1}(0) = \phi_l(t_{j+1}), \quad U^{j+1}(1) = \phi_r(t_{j+1}), \end{cases} \tag{3}$$

where, $\mathcal{L}_\varepsilon^{\Delta t} U^{j+1}(x) = -\varepsilon (U_{xx}(x))^{j+1} + p(x)(U_x(x))^{j+1} + q^{j+1}(x) U^{j+1}(x)$,

$$R^{j+1}(x) = \begin{cases} -r^{j+1}(x)\psi^{j+1}(x) + g^{j+1}(x) + \beta U^j(x) - \beta \sum_{k=1}^{j} b_k \Big(U^{j-k+1}(x) - U^{j-k}(x) \Big), \\ \qquad \text{for} \quad j = 1,2,...,n, \\ -r^{j+1}(x) U^{j-n+1}(x) + g^{j+1}(x) + \beta U^j(x) - \beta \sum_{k=1}^{j} b_k \Big(U^{j-k+1}(x) - U^{j-k}(x) \Big), \\ \qquad \text{for} \quad j = n+1, n+2, ..., N, \end{cases}$$

and $U^{j+1}(x)$ is the approximation to $u(x, t_{j+1})$.

The semi-discrete scheme (3) satisfy the following semi-discrete maximum principle.

Lemma 3 [22] *Let $\vartheta^{j+1}(x)$ be a sufficiently smooth function on the domain $[0, 1]$ satisfying $\vartheta^{j+1}(0) \geq 0$, $\vartheta^{j+1}(1) \geq 0$ and $\left(\beta + \mathcal{L}_\varepsilon^{\Delta t}\right) \vartheta^{j+1}(x) \geq 0, \quad \forall x \in [0,1]$. Then $\vartheta^{j+1}(x) \geq 0, \forall x \in [0,1]$.*

Lemma 4 [21]

The local truncation error $e_{\Delta t}^{j+1}$ in (2) is bounded.

$$\left| e_{\Delta t}^{j+1} \right| \leq C(\Delta t)^{2-\alpha}.$$

where C is constant independent of the perturbation parameter.

Lemma 5 (Ref, [14]) *The derivatives of the solution $U^{j+1}(x)$ of the semi-discrete problem (3) satisfies the bound*

$$\frac{d^m U^{j+1}(x)}{dx^m} \le C\left(1+\varepsilon^{-m}\exp\left(\frac{-p(1-x)}{\varepsilon}\right)\right),$$
$$x\in[0,1], \quad m=0,1,2,3,4.$$

Spatial discretization

Next, the domain along the spatial direction is discretized as: $\Omega_x = \{x_i : x_i = ih, i = 0,1,2,3,...,M\}$, where $h=\frac{1}{M}$ and M is the number of sub-intervals in [0, 1]. Then, on each sub-interval $[x_i, x_{i+1}]$, $i = 0,1,2,...,M-1$, define a non-polynomial cubic spline of the form:

$$S^{j+1}(x) = a_i\left(e^{\omega(x-x_i)}+e^{-\omega(x-x_i)}\right) + b_i\left(e^{\omega(x-x_i)}-e^{-\omega(x-x_i)}\right) + c_i(x-x_i)+d_i, \tag{4}$$

where a_i, b_i, c_i and d_i are unknown coefficients to be determined and ω is free parameter that is used to raise the accuracy of the method. To determine the unknowns in (4), define the following.

$$S^{j+1}(x_i) = U_i^{j+1}, \qquad S_{xx}^{j+1}(x_i) = M_i^{j+1},$$
$$S^{j+1}(x_{i+1}) = U_{i+1}^{j+1}, \qquad S_{xx}^{j+1}(x_{i+1}) = M_{i+1}^{j+1}. \tag{5}$$

Differentiating (4) successively, gives:

$$S_x^{j+1}(x) = \omega a_i\left(e^{\omega(x-x_i)}-e^{-\omega(x-x_i)}\right)+\omega b_i\left(e^{\omega(x-x_i)}+e^{-\omega(x-x_i)}\right)+c_i,$$
$$S_{xx}^{j+1}(x) = \omega^2 a_i\left(e^{\omega(x-x_i)}+e^{-\omega(x-x_i)}\right)+\omega^2 b_i\left(e^{\omega(x-x_i)}-e^{-\omega(x-x_i)}\right), \tag{6}$$

Now, using the relations in (5) into (4) and (6), we get:

$$a_i = \frac{M_i^{j+1}}{2\omega^2}, \qquad c_i = \frac{U_{i+1}^{j+1}-U_i^{j+1}}{h} - \frac{M_{i+1}^{j+1}-M_i^{j+1}}{\omega\theta},$$
$$b_i = \frac{2M_{i+1}^{j+1} - M_i^{j+1}\left(e^{\theta}+e^{-\theta}\right)}{2\omega^2\left(e^{\theta}-e^{-\theta}\right)}, \qquad d_i = U_i^{j+1} - \frac{M_i^{j+1}}{\omega^2}, \tag{7}$$

where $\omega h = \theta$.

But, the continuity of the first derivative for the non-polynomial cubic spline at $x = x_i$, gives:

$$\omega a_{i-1}\left(e^{\theta}-e^{-\theta}\right)+\omega b_{i-1}\left(e^{\theta}+e^{-\theta}\right)+c_{i-1} = 2\omega b_i + c_i. \tag{8}$$

Reducing the indexes of (7) by one and substituting into (8), gives:

$$\omega\left(\frac{M_{i-1}^{j+1}}{2\omega^2}\right)\left(e^{\theta}-e^{-\theta}\right)+\omega\left(\frac{2M_i^{j+1}-M_{i-1}^{j+1}(e^{\theta}+e^{-\theta})}{2\omega^2(e^{\theta}-e^{-\theta})}\right)\left(e^{\theta}+e^{-\theta}\right)+\frac{U_i^{j+1}-U_{i-1}^{j+1}}{h}-$$
$$\frac{M_i^{j+1}-M_{i-1}^{j+1}}{\omega\theta} = 2\omega\left(\frac{2M_{i+1}^{j+1}-M_i^{j+1}(e^{\theta}+e^{-\theta})}{2\omega^2(e^{\theta}-e^{-\theta})}\right)+\frac{U_{i+1}^{j+1}-U_i^{j+1}}{h}-\frac{M_{i+1}^{j+1}-M_i^{j+1}}{\omega\theta}.$$

Simplifying and rearranging gives:

$$\frac{U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}}{h^2}=\left(\frac{1}{\theta^2}-\frac{2}{\theta(e^{\theta}-e^{-\theta})}\right)M_{i-1}^{j+1}+\left(\frac{-2}{\theta^2}+\frac{2(e^{\theta}+e^{\theta})}{\theta(e^{\theta}-e^{-\theta})}\right)M_i^{j+1}+\left(\frac{1}{\theta^2}-\frac{2}{\theta(e^{\theta}-e^{-\theta})}\right)M_{i+1}^{j+1},$$

which can be rewritten as:

$$\frac{U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}}{h^2}=\lambda_1 M_{i-1}^{j+1}+2\lambda_2 M_i^{j+1}+\lambda_1 M_{i+1}^{j+1},\quad \text{for}\quad i=1,2,3,...,M-1, \tag{9}$$

where $\lambda_1=\frac{1}{\theta^2}-\frac{2}{\theta(e^{\theta}-e^{-\theta})}$ and $\lambda_2=\frac{-1}{\theta^2}+\frac{(e^{\theta}+e^{\theta})}{\theta(e^{\theta}-e^{-\theta})}$. Here, as $\theta\to 0$, $\lambda_1+\lambda_2\to\frac{1}{2}$ and their corresponding value is determined from the truncation error. That is, following the approach in [23], the truncation error in (9) is given by:

$$T_1(h)=\frac{h^4}{3}\left(-2\lambda_1+\lambda_2\right)p_i U_{xxx}^{j+1}(x_i)+h^4\left(1-12\lambda_1\right)\varepsilon U_{xxxx}^{j+1}(x_i)+O(h^6).$$

It is clear that, for the choice λ_1 and λ_2 whose sum $\frac{1}{2}$ is, $T_1(h)=O(h^4)$.

Considering the second order differential equation in (3) and using the relation in (5), corresponding to $M_i^{j+1}=S_{xx}^{j+1}(x_i)=U_{xx}^{j+1}(x_i)$, yields:

$$\begin{aligned}-\varepsilon M_i^{j+1}&=R_i^{j+1}-p_iU_x^{j+1}(x_i)-Q_i^{j+1}U_i^{j+1},\\ -\varepsilon M_{i-1}^{j+1}&=R_{i-1}^{j+1}-p_{i-1}U_x^{j+1}(x_{i-1})-Q_{i-1}^{j+1}U_{i-1}^{j+1},\\ -\varepsilon M_i^{j+1}&=R_{i+1}^{j+1}-p_{i+1}U_x^{j+1}(x_{i+1})-Q_{i+1}^{j+1}U_{i+1}^{j+1}.\end{aligned} \tag{10}$$

Substituting (10) into (9), gives:

$$\begin{aligned}&\frac{-\varepsilon}{h^2}\left(U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\right)\\&=\lambda_1\left(R_{i-1}^{j+1}-p_{i-1}U_x^{j+1}(x_{i-1})-Q_{i-1}^{j+1}U_{i-1}^{j+1}\right)+\\&2\lambda_2\left(R_i^{j+1}-p_iU_x^{j+1}(x_i)-Q_i^{j+1}U_i^{j+1}\right)\\&+\lambda_1\left(R_{i+1}^{j+1}-p_{i+1}U_x^{j+1}(x_{i+1})-Q_{i+1}^{j+1}U_{i+1}^{j+1}\right),\end{aligned}$$

where $Q_i^{j+1}=q_i^{j+1}+\beta$.

Consider the following finite difference approximations from [23].

$$\begin{aligned}U_x^{j+1}(x_i)&=\frac{U_{i+1}^{j+1}-U_{i-1}^{j+1}}{2h},\\ U_x^{j+1}(x_{i-1})&=\frac{-U_{i+1}^{j+1}+4U_i^{j+1}-3U_{i-1}^{j+1}}{2h},\\ U_x^{j+1}(x_{i+1})&=\frac{3U_{i+1}^{j+1}-4U_i^{j+1}+U_{i-1}^{j+1}}{2h}.\end{aligned} \tag{11}$$

Using the approximations in (11) into (10) and rearranging gives:

$$\begin{aligned}\mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1}=&\frac{-\varepsilon}{h^2}\left(U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\right)+\frac{\lambda_1 p_{i-1}}{2h}\left(-U_{i+1}^{j+1}+4U_i^{j+1}-3U_{i-1}^{j+1}\right)+\\&\frac{\lambda_2 p_i}{h}\left(U_{i+1}^{j+1}-U_{i-1}^{j+1}\right)+\frac{\lambda_1 p_{i+1}}{2h}\left(3U_{i+1}^{j+1}-4U_i^{j+1}+U_{i-1}^{j+1}\right)+\lambda_1 Q_{i-1}^{j+1}U_{i-1}^{j+1}+\\&2\lambda_2 Q_i^{j+1}U_i^{j+1}+\lambda_1 Q_{i+1}^{j+1}U_{i+1}^{j+1}=\lambda_1 R_{i-1}^{j+1}+2\lambda_2 R_i^{j}+\lambda_1 R_{i+1}^{j+1},\ \text{for}\quad i=1,2,...,M-1.\end{aligned} \tag{12}$$

Determination of the exponentially fitting factor

To overwhelm the effect of the perturbation parameter, we introduced an exponentially fitting factor σ to the term containing the perturbation parameter in (12) in the following form.

$$\frac{-\varepsilon\sigma}{h^2}\Big(U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\Big)+\frac{\lambda_1 p_{i-1}}{2h}\Big(-U_{i+1}^{j+1}+4U_i^{j+1}-3U_{i-1}^{j+1}\Big)+ \\ \frac{\lambda_2 p_i}{h}\Big(U_{i+1}^{j+1}-U_{i-1}^{j+1}\Big)+\frac{\lambda_1 p_{i+1}}{2h}\Big(3U_{i+1}^{j+1}-4U_i^{j+1}+U_{i-1}^{j+1}\Big)+\lambda_1 Q_{i-1}^{j+1}U_{i-1}^{j+1}+ \\ 2\lambda_2 Q_i^{j+1}U_i^{j+1}+\lambda_1 Q_{i+1}^{j+1}U_{i+1}^{j+1}=\lambda_1 R_{i-1}^{j+1}+2\lambda_2 R_i^{j}+\lambda_1 R_{i+1}^{j+1}. \tag{13}$$

Multiplying (13) by h and taking a limit as $h\to 0$ gives:

$$\frac{-\sigma}{\rho}\lim_{h\to 0}\Big(U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\Big)+\frac{\lambda_1 p(1)}{2}\lim_{h\to 0}\Big(-U_{i+1}^{j+1}+4U_i^{j+1}-3U_{i-1}^{j+1}\Big)+ \\ \lambda_2 p(1)\lim_{h\to 0}\Big(U_{i+1}^{j+1}-U_{i-1}^{j+1}\Big)+\frac{\lambda_1 p(1)}{2}\lim_{h\to 0}\Big(3U_{i+1}^{j+1}-4U_i^{j+1}+U_{i-1}^{j+1}\Big)=0, \tag{14}$$

where $\rho=\frac{h}{\varepsilon}$. But, from the theory of singular perturbation [24], taking the zero-order asymptotic solution $U^{j+1}(x)$ of (3) and expanding it using Taylor's series expansion about $x=0$ gives:

$$U^{j+1}(x)=U_0^{j+1}(x)+\Big(\phi_r(t_{j+1})-U_0^{j+1}(1)\Big)\exp\left(\frac{-p(1)(1-x)}{\varepsilon}\right)+O(\varepsilon),$$

where $U_0(x)$ is the solution of the reduced problem. Therefore, at $x=x_i$, we have:

$$\begin{cases} \lim_{h\to 0}\Big(U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\Big)=\Theta\Big(\exp(p(1)\rho)-2+\exp(-p(1)\rho)\Big), \\ \lim_{h\to 0}\Big(-U_{i+1}^{j+1}+4U_i^{j+1}-3U_{i-1}^{j+1}\Big)=\Theta\Big(-\exp(p(1)\rho)+4-3\exp(-p(1)\rho)\Big), \\ \lim_{h\to 0}\Big(3U_{i-1}^{j+1}-4U_i^{j+1}+U_{i+1}^{j+1}\Big)=\Theta\Big(3\exp(p(1)\rho)-4+\exp(-p(1)\rho)\Big), \\ \lim_{h\to 0}\Big(U_{i+1}^{j+1}-U_{i-1}^{j+1}\Big)=\Theta\Big(\exp(p(1)\rho)-\exp(-p(1)\rho)\Big), \end{cases} \tag{15}$$

where $\Theta=\Big(\phi_r(t_{j+1})-U_0^{j+1}(1)\Big)\exp(-p(1)(\frac{1}{\varepsilon}-i\rho))$. Using (15) into (14), simplifying, rearranging and adopting the result to a variable coefficient gives:

$$\sigma_i=\rho p_i(\lambda_1+\lambda_2)\coth\left(\frac{\rho p_i}{2}\right). \tag{16}$$

Following the approach in [21], we have:

$$\varepsilon\left(\rho p_i(\lambda_1+\lambda_2)\coth\left(\frac{\rho p_i}{2}\right)-1\right)\le\frac{h^2}{h+\varepsilon}. \tag{17}$$

The full discrete scheme

Now, inducing the exponential fitting factor obtained in (16) into (12), the full discrete scheme is given as:

$$\mathcal{L}_\varepsilon^{\Delta t,h}U_i^{j+1}=\lambda_1 R_{i-1}^{j+1}+2\lambda_2 R_i^{j+1}+\lambda_1 R_{i+1}^{j+1},\ \text{for}\quad i=1,2,...,M-1, \tag{18}$$

where,

$$R_i^{j+1}=\begin{cases} -r_i^{j+1}\psi_i^{j+1}+g_i^{j+1}+\beta U_i^j-\beta\sum_{k=1}^{j}b_k\Big(U_i^{j-k+1}-U_i^{j-k}\Big), & \text{for}\quad j=1,2,...,n, \\ -r_i^{j+1}U_i^{j-n+1}+g_i^{j+1}+\beta U_i^j-\beta\sum_{k=1}^{j}b_k\Big(U_i^{j-k+1}-U_i^{j-k}\Big), & \text{for}\ j=n+1,n+2,...,N, \end{cases}$$

$$\begin{aligned}\mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1} =&\frac{-\varepsilon\sigma_i}{h^2}\left(U_{i-1}^{j+1}-2U_i^{j+1}+U_{i+1}^{j+1}\right)+\frac{\lambda_1 p_{i-1}}{2h}\left(-U_{i+1}^{j+1}+4U_i^{j+1}-3U_{i-1}^{j+1}\right)+\\ &\frac{\lambda_2 p_i}{h}\left(U_{i+1}^{j+1}-U_{i-1}^{j+1}\right)+\frac{\lambda_1 p_{i+1}}{2h}\left(3U_{i+1}^{j+1}-4U_i^{j+1}+U_{i-1}^{j+1}\right)+\\ &\lambda_1 Q_{i-1}^{j+1}U_{i-1}^{j+1}+2\lambda_2 Q_i^{j+1}U_i^{j+1}+\lambda_1 Q_{i+1}^{j+1}U_{i+1}^{j+1},\end{aligned}$$

which can be rewritten in a three term recurrence relation as:

$$\begin{aligned}&F_i^- U_{i-1}^{j+1}+F_i^0 U_i^{j+1}+F_i^+ U_{i+1}^{j+1}=H_i^{j+1},\\ &\text{for} \quad i=1,2,3,...,M-1,\end{aligned} \tag{19}$$

where,

$$\begin{aligned}F_i^- &= \frac{-\varepsilon\sigma_i}{h^2}-\frac{3\lambda_1 p_{i-1}}{2h}-\frac{\lambda_2 p_i}{h}+\frac{\lambda_1 p_{i+1}}{2h}+\lambda_1 Q_{i-1}^{j+1},\\ F_i^0 &= \frac{2\varepsilon\sigma_i}{h^2}+\frac{2\lambda_1 p_{i-1}}{h}-\frac{2\lambda_1 p_{i+1}}{h}+2\lambda_2 Q_i^{j+1},\\ F_i^{j+1} &= \frac{-\varepsilon\sigma_i}{h^2}-\frac{\lambda_1 p_{i-1}}{2h}+\frac{\lambda_2 p_i}{h}+\frac{3\lambda_1 p_{i+1}}{2h}+\lambda_1 Q_{i+1}^{j+1},\\ H_i^{j+1} &= \lambda_1 R_{i-1}^{j+1}+2\lambda_2 R_i^{j+1}+\lambda_1 R_{i+1}^{j+1}.\end{aligned}$$

Stability and uniform convergence analysis

The following lemma guarantee the existence of a unique discrete solution for the scheme in (18).

Lemma 6 [24](*Discrete comparison principle*) *Suppose* $\mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1} \le \mathcal{L}_{\varepsilon}^{\Delta t,h}V_i^{j+1}$, *for* $1 \le i \le M-1$, *such that* $U_0^{j+1} \le V_0^{j+1}$ *and* $U_M^{j+1} \le V_M^{j+1}$. *Then,* $U_i^{j+1} \le V_i^{j+1}$, *for* $i=1(1)M$.

Lemma 7 *The solution* U_i^{j+1} *of the full discrete problem in* (18) *at each* $(j+1)^{th}$ *time level satisfies the bound*:

$$|U_i^{j+1}| \le \frac{||\mathcal{L}_{\varepsilon}^{h,\Delta t}U_i^{j+1}||}{\Upsilon}+\max\{|\phi_l(t_{j+1})|,|\phi_r(t_{j+1})|\}$$

where, $Q_i^{j+1} \ge \Upsilon > 0$.

Proof Define the barrier functions $\vartheta_{i,j+1}^{\pm}=\Pi \pm U_i^{j+1}$, where $\Pi=\frac{||\mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1}||}{\Upsilon}+\max\{|\phi_l(t_{j+1})|,|\phi_r(t_{j+1})|\}$. Then $\vartheta_{0,j+1}^{\pm}=\Pi \pm U_0^{j+1} \ge 0$, and $\vartheta_{M,j+1}^{\pm}=\Pi \pm U_M^{j+1} \ge 0$.

Moreover, for $1 \le i \le M-1$, we have

$$\begin{aligned}\mathcal{L}_{\varepsilon}^{\Delta t,h}\vartheta_{i,j+1}^{\pm} =&\frac{-\varepsilon\sigma_i}{h^2}\left(\vartheta_{i-1,j+1}^{\pm}-2\vartheta_{i,j+1}^{\pm}+\vartheta_{i+1,j+1}^{\pm}\right)+\frac{\lambda_1 p_{i-1}}{2h}\left(-\vartheta_{i+1,j+1}^{\pm}+4\vartheta_{i,j+1}^{\pm}-3\vartheta_{i-1,j+1}^{\pm}\right)+\\ &\frac{\lambda_2 p_i}{h}\left(\vartheta_{i+1,j+1}^{\pm}-\vartheta_{i-1,j+1}^{\pm}\right)+\frac{\lambda_1 p_{i+1}}{2h}\left(3\vartheta_{i+1,j+1}^{\pm}-4\vartheta_{i,j+1}^{\pm}+\vartheta_{i-1,j+1}^{\pm}\right)+\\ &\lambda_1 Q_{i-1}^{j+1}\vartheta_{i-1,j+1}^{\pm}+2\lambda_2 Q_i^{j+1}\vartheta_{i,j+1}^{\pm}+\lambda_1 Q_{i+1}^{j+1}\vartheta_{i+1,j+1}^{\pm},\\ =&\left(\lambda_1 Q_{i-1}^{j+1}+2\lambda_2 Q_i^{j+1}+\lambda_1 Q_{i+1}^{j+1}\right)\Pi \pm \mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1},\\ \ge&2\Upsilon(\lambda_1+\lambda_2)\Pi \pm \mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1}, \quad \text{since} \quad Q_i^{j+1} \ge \Upsilon > 0, \text{for } i=1(1)M-1,\\ \ge&\Upsilon\Pi \pm \mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1}, \quad \text{since} \quad \lambda_1+\lambda_2=\frac{1}{2},\\ \ge&\Upsilon\left(\frac{||\mathcal{L}_{\varepsilon}^{h,\Delta t}U_i^{j+1}||}{\Upsilon}+\max\{|\phi_l(t_{j+1})|,|\phi_r(t_{j+1})|\}\right) \pm \mathcal{L}_{\varepsilon}^{\Delta t,h}U_i^{j+1},\\ \ge&0.\end{aligned}$$

Then, the application of lemma 6 results $\vartheta^{\pm}_{i,j+1} \geq 0, \quad \forall i = 0(1)M$. Therefore, the desired bound hold. □

Lemma 8 (Ref [16]). *Let M be a fixed mesh number. Then, for* $\varepsilon \to 0$, *the following holds*:

$$\lim_{\varepsilon \to 0} \max_{1 \leq i \leq M-1} \varepsilon^{-m} \exp\left(\frac{-p(1-x_i)}{\varepsilon}\right) = 0, \quad m = 1, 2, 3, ...,$$

where $x_i = ih, \quad h = \frac{1}{M}, \quad \forall i = 1, 2, 3, ..., M-1.$

The following are an input in proving the next theorem. That is applying Taylor series expansion about $x = x_i$, we have:

$$\begin{aligned}
&\left| -\left(\frac{d^2}{dx^2} - \delta_x^2\right) U^{j+1}(x_i) \right| \leq Ch^2 \left| \frac{d^4 U^{j+1}(x_i)}{dx^4} \right|, \\
&\left| \frac{dU^{j+1}(x_{i-1})}{dx} - \left(\frac{-U_{i+1}^{j+1} + 4U_i^{j+1} - 3U_{i-1}^{j+1}}{2h} \right) \right| \leq Ch^2 \left| \frac{d^3 U^{j+1}(x_i)}{dx^3} \right|, \\
&\left| \frac{dU^{j+1}(x_{i+1})}{dx} - \left(\frac{3U_{i+1}^{j+1} - 4U_i^{j+1} + U_{i-1}^{j+1}}{2h} \right) \right| \leq Ch^2 \left| \frac{d^3 U^{j+1}(x_i)}{dx^3} \right|, \\
&\left| \left(\frac{d}{dx} - \delta_x^0\right) U^{j+1}(x_i) \right| \leq Ch^2 \left| \frac{d^3 U^{j+1}(x_i)}{dx^3} \right|,
\end{aligned} \tag{20}$$

Lemma 9 *Let* $U^{j+1}(x)$ *be the solution of the semi-discrete problem in* (3) *and* U_i^{j+1} *is the solution of the full discrete problem in* (18). *Then, the following error bound hold.*

$$\left| \mathcal{L}_\varepsilon^{h,\Delta t} \left(U^{j+1}(x_i) - U_i^{j+1} \right) \right| \leq \frac{Ch^2}{\varepsilon + h}.$$

Proof Consider the truncation error:

$$\begin{aligned}
\left| \mathcal{L}_\varepsilon^{h,\Delta t} \left(U^{j+1}(x_i) - U_i^{j+1} \right) \right| \leq &\Bigg| \varepsilon \left(\frac{d^2}{dx^2} - \sigma_i \delta_x^2 \right) U^{j+1}(x_i) + \\
&\lambda_1 p(x_{i-1}) \left(\frac{d}{dx} U^{j+1}(x_{i-1}) - \left(\frac{-3U_{i-1}^{j+1} + 4U_i^{j+1} - U_{i+1}^{j+1}}{2h} \right) \right) + \\
&2\lambda_2 p(x_i) \left(\frac{d}{dx} - \delta_x^0 \right) U^{j+1}(x_i) \Bigg| + \\
&\lambda_1 p(x_{i+1}) \left(\frac{d}{dx} U^{j+1}(x_{i+1}) - \left(\frac{3U_{i-1}^{j+1} - 4U_i^{j+1} + U_{i+1}^{j+1}}{2h} \right) \right) + \\
&\left| T_1(h) \right|.
\end{aligned}$$

Further, rearranging gives:

$$\begin{aligned}\left|\mathcal{L}_{\varepsilon}^{h,\Delta t}\left(U^{j+1}(x_i)-U_i^{j+1}\right)\right| \leq & \left|\varepsilon\left(\sigma_i-1\right)\delta_x^2U^{j+1}(x_i)\right|+\left|\varepsilon\left(\frac{d^2}{dx^2}-\delta_x^2\right)U^{j+1}(x_i)\right|+\\ & \left|\lambda_1 p(x_{i-1})\left(\frac{d}{dx}U^{j+1}(x_{i-1})-\left(\frac{-3U_{i-1}^{j+1}+4U_i^{j+1}-U_{i+1}^{j+1}}{2h}\right)\right)\right|+\\ & \left|2\lambda_2 p(x_i)\left(\frac{d}{dx}-\delta_x^0\right)U^{j+1}(x_i)\right|+\\ & \left|\lambda_1 p(x_{i+1})\left(\frac{d}{dx}U^{j+1}(x_{i+1})-\left(\frac{3U_{i-1}^{j+1}-4U_i^{j+1}+U_{i+1}^{j+1}}{2h}\right)\right)\right|+\\ & \left|T_1(h)\right|,\end{aligned}$$

Using the bounds in (17) and (20) gives:

$$\begin{aligned}\left|\mathcal{L}_{\varepsilon}^{h,\Delta t}\left(U^{j+1}(x_i)-U_i^{j+1}\right)\right| \leq & \frac{h^2}{\varepsilon+h}\left|\left|\frac{d^2U^{j+1}(x_i)}{dx^2}\right|\right|+C\varepsilon h^2\left|\left|\frac{d^4U^{j+1}(x_i)}{dx^4}\right|\right|+\\ & Ch^2\left|\left|\frac{d^3U^{j+1}(x_i)}{dx^3}\right|\right|+Ch^4\left(-2\lambda_1+\lambda_2\right)\left|\left|\frac{d^3U^{j+1}(x_i)}{dx^3}\right|\right|+\\ & C\varepsilon h^4\left(1-12\lambda_1\right)\left|\left|\frac{d^3U^{j+1}(x_i)}{dx^3}\right|\right|.\end{aligned}$$

The application of lemma 5 gives:

$$\begin{aligned}\left|\mathcal{L}_{\varepsilon}^{h,\Delta t}\left(U^{j+1}(x_i)-U_i^{j+1}\right)\right| \leq & \frac{Ch^2}{\varepsilon+h}\left(1+\varepsilon^{-2}\exp\left(\frac{-p(1-x_i)}{\varepsilon}\right)\right)+\\ & C\varepsilon h^2\left(1+\varepsilon^{-4}\exp\left(\frac{-p(1-x_i)}{\varepsilon}\right)\right)+\\ & Ch^2\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-x_i)}{\varepsilon}\right)\right)+\\ & Ch^4\left(-2\lambda_1+\lambda_2\right)\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-x_i)}{\varepsilon}\right)\right)+\\ & C\varepsilon h^4\left(1-12\lambda_1\right)\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-x_i)}{\varepsilon}\right)\right),\\ \leq & \frac{Ch^2}{\varepsilon+h}\left(1+\varepsilon^{-3}\exp\left(\frac{-p(1-x_i)}{\varepsilon}\right)\right), \quad \text{since} \quad \varepsilon^{-3}\geq\varepsilon^{-2}\end{aligned}$$

Therefore, the application of lemma 8 gives the required bound. □

Theorem 1 *The discrete solution U_i^{j+1} of the problem in* (18), *satisfies the following error bound.*

$$\sup_{0<\varepsilon\leq 1}\max_{x_i\in[0,1]}\left|U^{j+1}(x_i-U_i^{j+1})\right|\leq CM^{-1},$$

where C is a positive constant independent of the perturbation parameter ε.

Proof Note that from lemma 9, as $\varepsilon\to 0$, $\frac{h^2}{h+\varepsilon}\to CM^{-1}$, since $h=\frac{1}{M}$. Then, the application of lemma 6 in to lemma 9 gives the required bound. □

Here, whenever $\varepsilon>h$, the obtained method gives a second-order uniformly convergent. On the other hand, when $\varepsilon\ll h$ the method is first order uniformly convergent in the space direction.

Table 1 Comparison of maximum absolute error for Example 1 for a fixed $\varepsilon = 2^{-10}$, $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$ and different values of α

$\alpha \downarrow (M,N) \rightarrow$	(16, 20)	(32, 40)	(64, 80)	(128, 160)
Present Method				
0.25	5.6815e-03	3.2103e-03	1.6913e-03	8.3490e-04
0.5	5.9931e-03	3.3544e-03	1.7670e-03	8.7197e-04
0.75	6.4909e-03	3.5928e-03	1.8946e-03	9.3501e-04
Result in [20]				
0.25	1.1726e-02	6.3654e-02	3.3194E-02	1.6943e-03
0.5	1.2246e-02	6.6457e-03	3.4625e-03	1.7661e-03
0.75	1.3012e-02	7.0750e-03	3.6857e-03	1.8785e-03

Table 2 Comparison of maximum absolute error of Example 1 for a fixed $\alpha = 0.5$, $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$and different values of ε

$\varepsilon = 2^{-k} \downarrow / (M,N) \rightarrow$	(16,20)	(32,40)	(64,80)	(128,160)
Present method				
k=6	4.4305e-03	1.4685e-03	3.9700e-04	1.0233e-04
k=8	5.9888e-03	3.2323e-03	1.3387e-03	4.0971e-04
k=10	5.9931e-03	3.3544e-03	1.7670e-03	8.7197e-04
k=12	5.9931e-03	3.3544e-03	1.7682e-03	9.0458e-04
k=15	5.9931e-03	3.3544e-03	1.7682e-03	9.0458e-04
k=20	5.9931e-03	3.3544e-03	1.7682e-03	9.0458e-04
Result in [20]				
k=6	1.0088e-02	4.9401e-03	2.0143e-03	7.1385e-04
k=8	1.1863E-02	6.3546E-03	3.3404E-03	1.8221E-03
k=10	1.2246e-02	6.6457e-03	3.4625e-03	1.7661e-03
k=12	1.2336E-02	6.7141E-03	3.5082E-03	1.7930E-03
k=15	1.2361e-02	6.7337e-03	3.5212e-03	1.8011e-03
k=20	1.2365e-02	6.7364e-03	3.5230e-03	1.8022e-03

Theorem 2 *Let $u(x_i, t_j)$ be the solution of the problem (1) and U_i^{j+1} be the solution of the full discrete scheme (18). Then,*

Table 3 Maximum absolute error, uniform error and uniform rate of convergence of Example 2 for a fixed $\alpha = 0.5$, $\lambda_1 = \frac{1}{12}$, $\lambda_2 = \frac{5}{12}$ and different values of ε

$\varepsilon = 2^{-k} \downarrow / (M,N) \rightarrow$	(16,16)	(32,32)	(64,64)	(128,1128)	(256,256)
2^{-4}	2.7763e-03	7.0255e-04	1.8434e-04	4.9688e-05	1.3696e-05
2^{-6}	1.0220e-02	3.4939e-03	8.2988e-04	2.1462e-04	5.4950e-05
2^{-8}	1.2966e-02	7.1656e-03	3.1097e-03	9.6066e-04	2.2330e-04
2^{-10}	1.2973e-02	7.3879e-03	3.9419e-03	1.9756e-03	7.7086e-04
2^{-12}	1.2973e-02	7.3879e-03	3.9441e-03	2.0366e-03	1.0336e-03
2^{-14}	1.2973e-02	7.3879e-03	3.9441e-03	2.0366e-03	1.0341e-03
2^{-16}	1.2973e-02	7.3879e-03	3.9441e-03	2.0366e-03	1.0341e-03
$E^{M,N}$	1.2973e-02	7.3879e-03	3.9441e-03	2.0366e-03	1.0341e-03
$R^{N,M}$	0.8123	0.9055	0.9535	0.9778	-

$$\sup_{0<\varepsilon\leq 1} \left|\left| U(x_i) - U_i^{j+1} \right|\right|_D \leq C\left(M^{-1} + (\Delta t)^{2-\alpha}\right).$$

Proof The result of this theorem holds form the triangular property of norm, the error bounds in lemma 4 and theorem 1. □

Numerical result and discussion

To validate the main result of our work, the following three model examples are considered.

Example 1 We considered the following time-fractional singularly perturbed problem which is taken from [22]:

$$\begin{cases} D^\alpha u(x,t) - \varepsilon u_{xx}(x,t) + (2-x^2)u_x(x,t) + (x+1)(t+1)u(x,t) = u(x,t-1) + 10t^2\exp(-t)x(1-x), \\ \qquad\qquad \text{for,} \quad (x,t) \in (0,1)\times(0,2], \\ u(x,t) = 0, \quad \text{for } (x,t) \in [0,1]\times[-1,0], \\ u(0,t) = 0, \quad u(1,t) = 0, \quad \text{for } t \in [0,2]. \end{cases}$$

Example 2 We considered the following time-fractional singularly perturbed problem which is taken from [22]:

$$\begin{cases} D^{\alpha}u(x,t) - \varepsilon u_{xx}(x,t) + (2-x^2)u_x(x,t) + xu(x,t) = u(x,t-1) + 10t^2\exp(-t)x(1-x), \\ \qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad \text{for,} \quad (x,t) \in (0,1)\times(0,2], \\ u(x,t) = 0, \quad \text{for } (x,t) \in [0,1]\times[-1,0], \\ u(0,t) = 0, \quad u(1,t) = 0, \quad \text{for } t \in [0,2]. \end{cases}$$

Example 3 We considered the following time-fractional singularly perturbed problem which is taken from [22]:

$$\begin{cases} D^{\alpha}u(x,t) - u_{xx}(x,t) + u_x(x,t) = \frac{2t^{2-\alpha}}{\Gamma(3-\alpha)} + 2x - 2, \qquad \text{for,} \quad (x,t) \in (0,1)\times(0,1], \\ u(x,0) = x^2, \quad \text{for } x \in [0,1], \\ u(0,t) = t^2, \quad u(1,t) = 1 + t^2, \quad \text{for } t \in [0,1]. \end{cases}$$

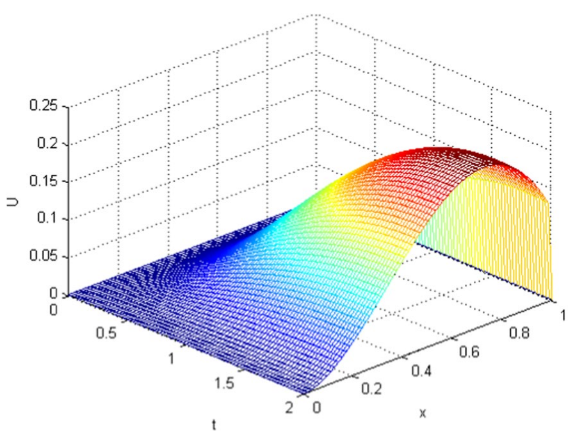

Fig. 1 Surface plot of Example 1 for $\varepsilon = 2^{-12}, \alpha = 0.5, M = 64$ and $N = 80$

Table 4 Comparison of maximum absolute error for Example 3 for a fixed $\varepsilon = 1, \lambda_1 = \frac{1}{12}, \lambda_2 = \frac{5}{12}$ and different values of α and

$\alpha \downarrow (M,N) \rightarrow$	(16, 16)	(32, 32)	(64, 64)	(128, 128)
Present method				
0.25	6.2801e-04	1.6545e-04	4.3996e-05	1.1828e-05
0.5	1.0742e-03	3.4815e-04	1.3367e-04	5.3069e-05
0.75	2.6933e-03	1.0857e-03	4.4332e-04	1.9557e-04
Result in [20]				
0.25	7.4538E-04	2.2828E-04	6.9461E-05	2.1047E-05
0.5	2.3962E-03	8.5395E-04	3.0333E-04	1.0759E-04
0.75	6.3731E-03	2.6801E-03	1.1263E-03	4.7338E-04

The exact solution of the first two model examples is not known. As a result, double mesh principle is applied to compute the maximum absolute error using:

$$E_{\varepsilon}^{M,N} = \max_{i,j} \left| U^{M,N}(x_i,t_j) - U^{2M,2N}(x_{2i},t_{2j}) \right|,$$

and the corresponding uniform error estimate is obtained by: $E^{M,N} = \max_{\varepsilon}(E_{\varepsilon}^{M,N})$. Finally, the uniform rate of convergence is calculated using:

$$R^{M,N} = \frac{\log(E^{M,N}) - \log(E^{2M,2N})}{\log 2}.$$

However, the exact solution of the third model example is known and it is given by $u(x,t) = x^2 + t^2$. Hence, the maximum absolute error is found by using:

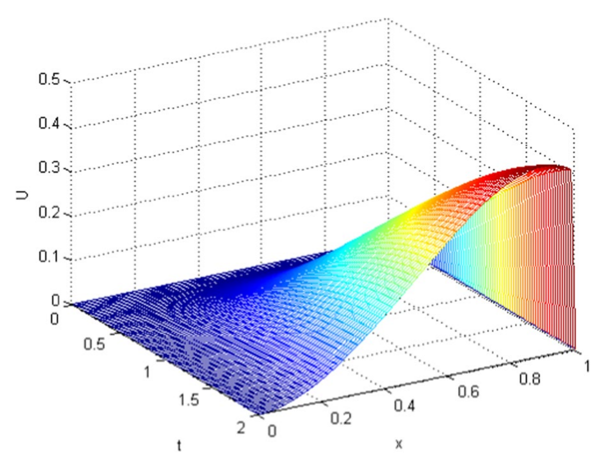

Fig. 2 Surface plot of Example 2 for $\varepsilon = 2^{-12}, \alpha = 0.5, M = 64$ and $N = 80$

$$E_{\varepsilon}^{M,N} = \max_{i,j} \left| u^{M,N}(x_i, t_j) - U^{M,N}(x_i, t_j) \right|,$$

where, $u(x, t)$ is the exact solution.

The first two model examples involves an arbitrary fractional order derivative α, a large temporal lag of magnitude $\wp = 1$ and a singular perturbation parameter ε multiplying the term containing highest derivative. Due the presence of ε, the solutions of the considered model examples exhibit a strong boundary layer at the right end point of the spatial domain whenever the value of ε approaches to 0 as it is depicted in figures 1 and 2, respectively, for example 1 and 2. The large temporal lag doesn't have an effect on the position and size of the boundary layer since the layer occur along the spatial domain only.

The maximum absolute error of the proposed method for each example is computed taking $\lambda_1 = \frac{1}{12}$ and $\lambda_2 = \frac{5}{12}$. The comparison of maximum absolute error of Example 1 for the present method with the method developed in [20], with a fixed value of ε and different values of α, is presented in Table 1. The result in this table depict that, the proposed method is more accurate than, the method in the literature. The comparison of the proposed method with the method in [20] in maximum absolute error, for Example 1 with a different values of ε and a fixed α is presented in Table 2. Again the result in this table also indicate that, the proposed method is convergent and more accurate than the result presented in [20]. The numerical result in Table 3 also indicate the maximum absolute error of Example 2 for different values of the perturbation parameter ε and a fixed value of α. From the result in this table, as the perturbation goes smaller and smaller, the maximum absolute error of the proposed method becomes stable and identical after showing some grow up indicating that, the proposed method is ϵ-uniform or uniformly convergent. Again, from the last rows of this table, we can clearly observe that, the proposed method is first order which is in agreement with the theoretical expectation. The comparison of maximum absolute error of Example 3 for the present method, with the method developed in [20], with a fixed value $\varepsilon = 1$ and different values of α, is presented in Table 4. The result in this table shows that the proposed method is more accurate than the result found in the literature. Figure 1 and Fig. 2 shows the solution profile of Example 1 and Example 2 for $\varepsilon = 2^{-12}$, $M = 64$, $N = 80$ and $\alpha = 0.5$, respectively. From the figures, one can observe that the numerical solution of the governing problem forms a strong right boundary layer as the perturbation becomes smaller and smaller.

Conclusion

In this paper, an exponentially fitted non-polynomial cubic spline method is proposed for solving time-fractional singularly perturbed convection-diffusion problems involving large temporal lag. The time-fractional derivative is considered in the Caputo sense and discretized uniformly using the implicit finite difference techniques. Then, an exponentially fitted non-polynomial cubic spline method is constructed along the spatial domain on a uniform mesh discretization. The ε-uniform convergence of the proposed method has rigorously proved and shown to be convergent with order of convergence $O((\Delta t)^{2-\alpha} + M^{-1})$. The proposed method is validated using two model examples and the experimental result is in agreement with the theoretical expectation. Moreover, the proposed method gives more accurate solution than some recent existing methods.

Limitations

The developed method is not layer resolving method since there is no sufficient number of mesh points in the boundary layer region.

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